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The Contact Polytope of the Leech Lattice

Mathieu Dutour Sikirić · Achill Schürmann · Frank Vallentin

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Abstract The contact polytope of a lattice is the convex hull of its shortest vectors. In this paper we classify the facets of the contact polytope of the Leech lattice up to symmetry. There are 1, 197, 362, 269, 604, 214, 277, 200 many facets in 232 orbits.

Keywords Leech lattice · Contact polytope · Conway groups · Voronoi cell

1 Introduction

An n-dimensional *lattice* L is a discrete subgroup of the n-dimensional Euclidean space $\mathbb{R}^n$ of the form $L = \{\sum_{i=1}^{n} \alpha_i b_i : \alpha_1, \ldots, \alpha_n \in \mathbb{Z}\}$, where $b_1, \ldots, b_n$ is a basis of $\mathbb{R}^n$. By $\lambda(L)$ we denote the Euclidean length of nonzero shortest vectors of L, and we denote by $\operatorname{Min} L$ the set of *shortest vectors*.

Every lattice comes with two important polytopes: The *contact polytope* of L is the convex hull of its shortest vectors

$$\mathrm{C}(L) = \operatorname{conv}\bigl\{v : v \in \operatorname{Min} L\bigr\},$$

M. Dutour Sikirić
Rudjer Bosković Institute, Bijenicka 54, 10000 Zagreb, Croatia
e-mail: mdsikir@irb.hr

A. Schürmann · F. Vallentin (✉)
Delft Institute of Applied Mathematics, Technical University of Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands
e-mail: f.vallentin@tudelft.nl

A. Schürmann
e-mail: a.schurmann@tudelft.nl

and the *Voronoi cell* of L is the region of points that are closer to the origin than to other lattice points,

$$\mathrm{V}(L) = \left\{ x \in \mathbb{R}^n : x \cdot v \leq \frac{1}{2} v \cdot v \text{ for all } v \in L \right\}.$$

Maybe one of the most remarkable lattices is the 24-dimensional Leech lattice Λ_{24}. It has 196,560 shortest vectors which is the highest possible number in dimension 24. Its *orthogonal group*, i.e., the group of orthogonal transformations preserving the lattice is the Conway group Co_0. It has $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 =$ 8,315,553,613,086,720,000 elements and is connected to many sporadic simple groups. We refer to the book [6] by Conway and Sloane for an extensive treatment of the Leech lattice.

Borcherds, Conway, Parker, Queen, Sloane [6, Chaps. 23 and 25] determine the vertices of the Voronoi cell of the Leech lattice. The Voronoi cell tiles the space $\mathbb{R}^n$ by translations; this gives the *Voronoi cell tiling* of $\mathbb{R}^n$. So, in the context of the Voronoi cell it is natural to consider orbits under the *isometry group* (the group generated by the orthogonal group of the Leech lattice together with lattice translations) acting on the Voronoi cell tiling. We denote the isometry group of the Leech lattice by Co_∞. There are 307 orbits of vertices in the Voronoi cell tiling under the action of Co_∞.

In this paper we determine the facets and their incidence relations of the contact polytope of the Leech lattice. We get the following result.

Theorem 1 *There are* 232 *orbits of facets of* $\mathrm{C}(\Lambda_{24})$ *under* Co_0.

The contact polytope and the Voronoi cell are related. To see this relation, we consider

$$\mathrm{C}(L)^* = \left\{ x \in \mathbb{R}^n : x \cdot v \leq \frac{1}{2} v \cdot v \text{ for all } v \in \operatorname{Min} L \right\},$$

which is the standard polar polytope scaled by a factor of $\frac{1}{2}\lambda(L)^2$. The faces of $\mathrm{C}(L)$ and of $\mathrm{C}(L)^*$ are in bijection. The bijection reverses the inclusion relation: k-dimensional faces of $\mathrm{C}(L)$ correspond to $(n-k)$-dimensional faces of $\mathrm{C}(L)^*$. In particular, vertices of $\mathrm{C}(L)^*$ correspond to facets of $\mathrm{C}(L)$. For these notions, we refer to the standard literature on polytope theory, e.g., the book by Ziegler [21].

We chose the scaling in the definition of $\mathrm{C}(L)^*$ so that it contains $\mathrm{V}(L)$. In the case of the Leech lattice some vertices of $\mathrm{V}(\Lambda_{24})$ and $\mathrm{C}(\Lambda_{24})^*$ are shared. As a side remark: One has the equality $\mathrm{C}(L)^* = \mathrm{V}(L)$ if and only if L is a root lattice, see Rajan and Shende [18].

Theorem 2 164 *orbits of vertices of* $\mathrm{C}(\Lambda_{24})^*$ *are also orbits of vertices of* $\mathrm{V}(\Lambda_{24})$. *They are listed in Table* 1 *in the complete version of the paper* [10]. *The additional* 68 *orbits of vertices are listed in Table* 2 *of* [10].

We classify the shared vertices in Sect. 2 and give them in Table 1 of [10]. In Sect. 3 we classify the additional vertices of $\mathrm{C}(\Lambda_{24})^*$ that are not vertices of $\mathrm{V}(\Lambda_{24})$.

We conclude the paper by Sect. 4, where we briefly explain our computational techniques.

The data presented here is also electronically available from [8].

2 Shared Vertices

In this section we explain the notation used in Table 1 of [10], which contains the 164 orbits of shared vertices mentioned in Theorem 2.

The vertices of the Voronoi cell of a lattice are centers of *empty spheres*, i.e., spheres $S(x, \|x\|)$ with center x and radius $\|x\|$ which contain lattice points on their boundary but not in their interior. The convex hull of lattice points on the boundary of such an empty sphere is called the *Delone cell* of the vertex x.

The Delone cells of the Leech lattice are classified by Borcherds, Conway, Parker, Queen, Sloane [6, Chaps. 23 and 25] up to the action of the isometry group Co_∞. For this classification, they use Coxeter–Dynkin diagrams.

A *Coxeter–Dynkin diagram* with vertex-set $\{1, \dots, N\}$ is a symmetric $N \times N$ matrix $(m_{ij})_{1\le i,j\le N}$ with ones on the diagonal and $m_{ij} \ge 2$ if $i \ne j$ and $m_{ij} \in \mathbb{N} \cup \{\infty\}$.

A Coxeter–Dynkin diagram is called *simply laced* if $m_{ij} = 2$, 3, or ∞. The *Cartan matrix* of a Coxeter–Dynkin diagram $(m_{ij})_{1\le i,j\le N}$ is the matrix $M = (-\cos\frac{\pi}{m_{ij}})_{1\le i,j\le N}$. A Coxeter–Dynkin diagram is called *spherical* if its Cartan matrix is positive definite and *affine* if its Cartan matrix is positive semidefinite. A Coxeter–Dynkin diagram is called *decomposable* if we can partition its vertex-set into $S_1 \cup S_2$ with $m_{ij} = 2$ if $i \in S_1$ and $j \in S_2$. It is called *indecomposable* otherwise. A Coxeter–Dynkin diagram D admits a unique decomposition into indecomposable Coxeter–Dynkin diagrams $D_1, \dots, D_r$, which we write as $D = D_1 D_2 \dots D_r$. The classification of spherical and affine Coxeter–Dynkin diagrams is presented, for example, in Humphreys [13, Sects. 2.4 and 4.7]. Here the famous $A - D - E$ diagrams show up, explained, e.g., by Hazewinkel, Hesselink, Siersma, and Veldkamp [12]. The spherical, simply laced, indecomposable Coxeter–Dynkin diagrams are a_n for $n \ge 1$, d_n for $n \ge 4$, and e_n for $6 \le n \le 8$. Each diagram corresponds to an indecomposable affine diagram: A_n, D_n, and E_n. All these diagrams are pictured, e.g., in [6, Fig. 23.1].

In the Leech lattice, a Coxeter–Dynkin diagram $(m_{ij})_{1\le i,j\le N}$ can be associated with a Delone cell with vertex-set $\{v_1, \dots, v_N\}$ by

$$m_{ij} = \begin{cases} 1 & \text{if } \|v_i - v_j\|^2 = 0, \\ 2 & \text{if } \|v_i - v_j\|^2 = 4, \\ 3 & \text{if } \|v_i - v_j\|^2 = 6, \\ \infty & \text{if } \|v_i - v_j\|^2 = 8. \end{cases}$$

As can been seen in Table 1 of [10], different Delone cells may have the same Coxeter–Dynkin diagram.

In Table 1 of [10] the rows are sorted first by the squared length $\|v\|^2$ (third column) of the vertex v. Second, they are sorted by the size of the stabilizer of v within the orthogonal group of the Leech lattice (fifth column) and then by the number of incident facets of $C(\Lambda_{24})^*$ (fourth column).

In the second column we give the Coxeter–Dynkin diagrams of the associated Delone cell of v. Note that the diagrams are affine if and only if the squared length of v equals 2, the maximum among shared vertices. In all other cases they are spherical. Furthermore, in the spherical cases the number of incident facets is always equal to the minimum possible number of 24. These observations follow from [6, Chaps. 23 and 25].

In the last column we give the MOG (Miracle Octad Generator) coordinates of representatives of each orbit which one has to multiply with α (sixth column). The MOG coordinates form a standard coordinate system for the Leech lattice. They are explained in [6, Chap. 11].

There are 307 orbits of vertices in the Voronoi cell tiling under the action of the isometry group Co_∞ of the Leech lattice. Our computation shows that there are 5,297 orbits of vertices of the single Voronoi cell $V(\Lambda_{24})$ under the action of the smaller, finite orthogonal group of the Leech lattice; 164 of them are shared with $C(\Lambda_{24})^*$.

3 Additional Vertices

There are 68 additional orbits of vertices of $C(\Lambda_{24})^*$ that are not vertices of the Voronoi cell of the Leech lattice. These additional vertices are characterized by the fact that the distance to a closest lattice point is strictly less than the distance $\|v\|$ to the origin.

Table 2 of [10] describes these 68 orbits. Like in Table 1 of [10], the rows are sorted (in this order) by the squared length $\|v\|^2$ (third column), the size of the stabilizer of v within the orthogonal group of the Leech lattice (fifth column), and then by the number of incident facets (fourth column).

In the second column we give names for diagrams. The first row corresponds to an exceptional vertex which we explain below. The other 67 rows correspond to graphs which we define later in Sect. 3.2.

3.1 The Exceptional Vertex

The first orbit of vertices is exceptional: Its squared norm $8/3 = 2.666\ldots$ is substantially bigger than the squared norm of all other vertices which lie in the interval $[1.92, 2.25]$. Its incidence number of 552 and the size of its stabilizer, which is the Conway group Co_3, are also substantially larger than the values for the other vertices. This orbit of vertices is a scaled copy of the vectors of Λ_{24}, having Euclidean norm $\sqrt{6}$.

In the contact polytope $C(\Lambda_{24})$ this exceptional vertex corresponds to a facet. Since it has maximum norm among all vertices, the corresponding facet is closest to the origin and has the largest possible circumsphere among all other facets of $C(\Lambda_{24})$. This solves a conjecture of Ballinger, Blekherman, Cohn, Giansiracusa, Kelly, and Schürmann [2, Sect. 3.7]. We note that a similar calculation as the one presented here, solves the corresponding conjecture about the contact polytope of the 23-dimensional lattice O_{23}, the shorter Leech lattice, which has 4,600 vertices.

The 23-dimensional point configuration, given by the 552 shortest vectors of the Leech lattice defining facets incident to the exceptional vertex, appears in several different contexts: It is universally optimal (Cohn and Kumar [5]), it defines 276 equiangular lines (Lemmens and Seidel [15]), and it defines an extreme Delone cell (Deza and Laurent [14, Chap. 16.3]). Moreover, it contains a wealth of remarkable substructures (see Cohn et al. [2]), e.g., the highly-symmetric point configurations discussed in the next section, and also others, e.g., the one defined by the McLaughlin graph.

3.2 The Other Vertices

To the remaining 67 orbits of vertices we associate a diagram as follows. Let v be one of these vertices, and let $w_1, \ldots, w_N$ be shortest vectors of the Leech lattice defining facets incident to v. Only the two inner products 1 and 2 occur between distinct vectors w_i and w_j. So we can define a graph with vertex-set $\{1, \ldots, N\}$ and edge-set $\{\{i, j\} : w_i \cdot w_j = 1\}$; the other inner product 2 defines nonedges.

Here again the graphs decompose into connected components where several of these occurring components are highly symmetric and have been studied in other contexts. We discuss them below; the graphs a_n, d_n, and e_n are already described in the previous section, and the remaining ones are in Fig. 1.

The *Higman–Sims graph* HS_{100} is the unique strongly regular graph with parameters $(100, 22, 0, 6)$. See Brouwer, Cohen, and Neumaier [4, Chap. 13.1].

The *Hoffman–Singleton graph* HS_{50} is the unique strongly regular graph with parameters $(50, 7, 0, 1)$. See [4, Chap. 13.1].

For the *Johnson graph* $J(7, 4)$, see [4, Chap. 9.1].

A (k, g)*-cage* is a regular graph of valency k and girth g which attains the minimum possible number of vertices. The $(5, 6)$-cage (incidence graph of a projective plane $\mathrm{PG}(2, 4)$) and the $(3, 8)$-cage (*Tutte–Coxeter graph*) are unique. See [4, Chap. 6.9] and Tutte [20].

The *Coxeter graph* Cox is the unique distance regular graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$. See [4, Chap. 12.3].

In Fig. 1 we list the remaining graphs. The vertices of these graphs only have degree one (white circles), degree two (sitting on edges, which are not depicted; see below), or degree three (black circles). We have three kinds of trees: $T_b^a c$ having $a + b + c + 4$ vertices, $T_b^a c_e^d$ having $a + b + c + d + e + 6$ vertices, and $T_b^a c^d e_g^f$ having $a + b + c + d + e + f + g + 8$ vertices; we have 12 other graphs $G_{n,m}$ with n vertices and m edges. In Fig. 1 the numbers on the edges show how many vertices of degree 2 sit on them, but in the following four cases we did not put these numbers: The graph $G_{24,30}$ has one vertex of degree 2 on every edge, $G_{25,30}$ is the Petersen graph which has one vertex of degree 2 on every edge, $G_{22,22}$ has three vertices of degree 2 on every edge, and the graph $G_{24,27}$ is the complete bipartite graph $K_{3,3}$ which has two vertices of degree 2 on every edge.

4 Computational Techniques

Computing the vertices of $\mathrm{C}(\Lambda_{24})^*$ from its facets is called a *polyhedral representation conversion problem*. A direct application of standard programs like Fukuda's

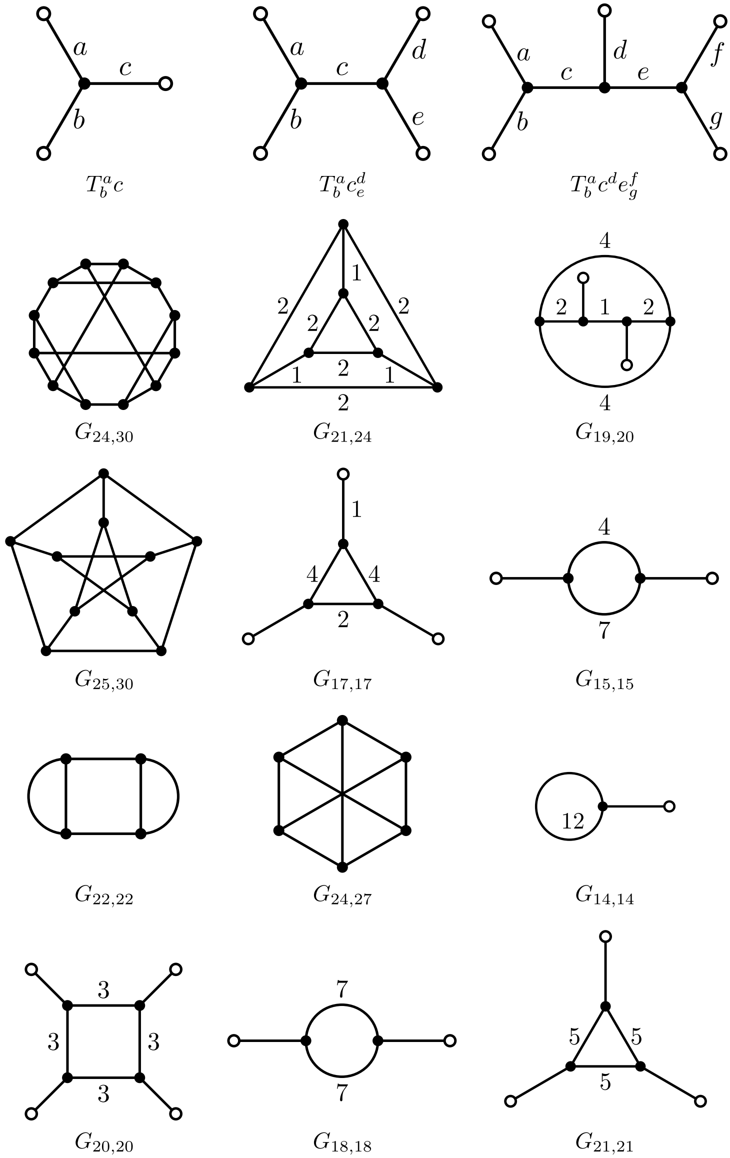

Fig. 1 Diagrams

`cdd` [11] or Avis' `lrs` [1] for this conversion is not feasible due to the extremely large number of vertices.

In order to exploit the symmetries of $\mathrm{C}(\Lambda_{24})^*$, we use the *adjacency decomposition method* which is surveyed in Bremner, Dutour Sikirić, and Schürmann [3]. An implementation by the first author is available from [7].

The adjacency decomposition method computes a complete list of inequivalent vertex representatives. First, one computes an initial vertex by solving a linear program and inserts it into the list of orbit representatives. From any such representative, we compute the list of adjacent vertices, and if they give a new orbit, we insert it into the list of representatives. After finitely many steps all orbits have been treated. Computing adjacent vertices is a lower-dimensional representation conversion problem. So this method can be applied recursively.

For $\mathrm{C}(\Lambda_{24})^*$, we had to come up with two case-specific insights:

From [2] it is known that the exceptional vertex of Sect. 3.1 is indeed a vertex of $\mathrm{C}(\Lambda_{24})^*$. We used it as starting vertex of the adjacency decomposition method.

For checking isomorphy and for computing stabilizers, we used the following standard strategy: we characterize a vertex of $\mathrm{C}(\Lambda_{24})^*$ by the set of its incident facets, and we represent the symmetry group Co_0 as a permutation group acting on the 196,560 shortest vectors of the Leech lattice. Then, we use the backtracking algorithm by Leon [16, 17] implemented in [19]. This worked reasonably fast for all the cases except for the two orbits of vertices having the same Coxeter–Dynkin diagram a_1^{25}. The stabilizer of the corresponding Delone cell under the isometry group Co_∞ is the Mathieu group M_{24}. Under the action of M_{24}, the 25 vertices of the Delone cell split into two orbits of size 1 and 24. Hence, these two orbits correspond to two distinct orbits of vertices of $\mathrm{C}(\Lambda_{24})^*$, one having stabilizer M_{24} and the other having stabilizer M_{23}. The backtracking algorithm of GAP could not decide in reasonable time whether or not two vertices with the same Coxeter–Dynkin diagram a_1^{25} are in the same orbit. So we used the third method of Sect. 3.5 of [9] to resolve this problem.

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